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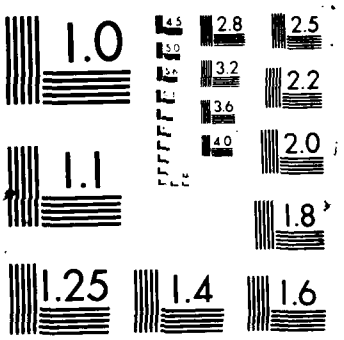
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The set of all bivariate distributions with support contained in {(i,j); i=1,2 and j = 1,2,...,n} which are positive quadrant dependent is a convex set. In this paper, an algebraic method is presented for the enumeration of all extreme points of the convex set. Certain measures of dependence, including Kendall's tau, are shown to be affine functions of convex set. This property of being affine helps us to evaluate the asymptotic power of tests based on these measures of dependence for testing the hypothesis of independence against strict positive quadrant dependence.

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by

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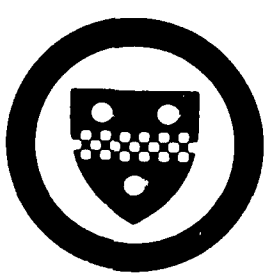

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ON THE EXTREME POINTS OF THE SET OF ALL 2xn BIVARIATE POSITIVE QUADRANT DEPENDENT DISTRIBUTIONS WITH FIXED MARGINALS AND SOME APPLICATIONS

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SUMMARY

The set of all bivariate distributions with support contained in $\{(i,j);\ i = 1,2 \text{ and } j = 1,2,\cdots,n\}$ which are positive quadrant dependent is a convex set. In this paper, an algebraic method is presented for the enumeration of all extreme points of this convex set. Certain measures of dependence, including Kendall's tau, are shown to be affine functions on this convex set. This property of being affine helps us to evaluate the asymptotic power of tests based on these measures of dependence for testing the hypothesis of independence against strict positive quadrant dependence.

Keywords : POSITIVE QUADRANT DEPENDENCE; CONVEX SET; EXTREME POINTS; MEASURES OF DEPENDENCE; KENDALL'S TAU; SOMER'S d; PEARSON'S CORRELATION COEFFICIENT; SPEARMAN'S RHO; HYPOTHESIS OF INDEPENDENCE; POWER OF A TEST; ASYMPTOTICS

1. INTRODUCTION AND PRELIMINARIES

A good understanding of the nature of dependence among multivariate probability distributions is useful for modelling multivariate random phenomenon. There is a multitude of dependence notions available in the literature. For a good account of these notions, one may refer to Lehmann (1966), Barlow and Proschan (1981), and Eaton (1982). In this paper, we concentrate on one particular notion of dependence, namely, that of positive quadrant dependence among discrete bivariate distributions and examine the structure of such distributions by performing an extreme point analysis. We now describe the problem. Let X and Y be two random variables with some joint probability distribution function F.

Assume that X takes only two values 1 and 2, and Y takes n values $1,2,\cdots,n$. The joint probability distribution of X and Y can be described by a matrix $P = (p_{ij})$ of order $2xn$, where $p_{ij} = Pr(X = i, Y = j)$, $i = 1,2$ and $j = 1,2,\cdots, n$. The random variables X and Y are said to be positive quadrant dependent (equivalently, F or P is said to be positive quadrant dependent) (PQD) if

$$Pr(X \leq i, Y \leq j) \geq Pr(X \leq i)\, Pr(Y \leq j)$$

for all i and j. Let $p_i = Pr(X = i)$, $i = 1,2$ and $q_j = Pr(Y = j)$, $j = 1,2, \cdots,n$. The above condition can be phrased, equivalently, as follows.

$$\begin{aligned} & p_{11} \geq p_1 q_1 \\ & p_{11} + p_{12} \geq p_1(q_1 + q_2) \\ & p_{11} + p_{12} + p_{13} \geq p_1(q_1 + q_2 + q_3) \\ & \cdots \qquad \cdots \qquad \cdots \qquad \cdots \\ & p_{11} + p_{12} + \cdots + p_{1n-1} \geq p_1(q_1 + q_2 + \cdots + q_{n-1}). \end{aligned} \tag{1.1}$$

Let M_{PQD} denote the collection of all bivariate positive quadrant dependent distributions with support contained in $\{(i,j)\ ;\ i = 1,2 \text{ and } j = 1,2,\cdots,n\}$. This set is not a convex set. An example of two bivariate distributions P_1 and P_2 in M_{PQD} and a number $0 < \lambda < 1$ such that $\lambda P_1 + (1-\lambda)P_2 \notin M_{PQD}$ are easy to find. However, if we fix the marginal distributions of X and Y, the above set becomes a convex set. More precisely, let p_1 and p_2, and $q_1, q_2, \ldots, q_n$ be two sets of non-negative numbers satisfying $p_1 + p_2 = 1 = q_1 + q_2 + \ldots + q_n$. Let

$$\begin{aligned} M_{PQD}(p_1,p_2;q_1,q_2,\ldots,q_n) = \{ & P = (p_{ij}) \in M_{PQD};\ p_{1j} + p_{2j} = q_j \text{ for} \\ & j = 1,2,\ldots,n \text{ and } p_{i1} + p_{i2} + \cdots + p_{in} \\ & = p_i \text{ for } i = 1,2\}. \end{aligned}$$

Under the bivariate distribution P in $M_{PQD}(p_1,p_2;q_1,q_2,\cdots,q_n)$, X and Y are positive quadrant dependent, X has the marginal distribution p_1, p_2, and Y has the marginal distribution $q_1,q_2,\cdots,q_n$. This set has nice properties.

Theorem 1. The set $M_{PQD}(p_1,p_2;q_1,q_2,\cdots,q_n)$ is compact and convex. More strongly, this set is a simplex.

Proof. The convexity follows from the inequalities (1.1). That the set is a simplex is obvious.

Even though the set M_{PQD} is not convex, it can be written as a union of compact convex sets, i.e.,

$$M_{PQD} = \cup M_{PQD}(p_1,p_2;q_1,q_2,\cdots,q_n),$$

where the union is taken over all $p_1,p_2;\ q_1,q_2,\cdots,q_n$ of non-negative numbers with $p_1 + p_2 = 1 = q_1 + q_2 + \cdots + q_n$. The fact that $M_{PQD}(p_1,p_2;q_1,q_2,\cdots,q_n)$ is a simplex implies that it has only a finite number of extreme points, and that every member of this set can be written as a convex combination of its extreme points. Looking at the notion of positive quadrant dependence from a global point of view as enunciated above is useful. Some of the applications given in Section 3 amplify this point. A good understanding of the nature of extreme points will provide a deep insight into the mechanism of positive quadrant dependence of two random variables.

One of the goals of this paper is to present an algebraic method to find all the extreme points of the convex set $M_{PQD}(p_1,p_2;q_1,q_2,\cdots,q_n)$. The cases $n = 2$ and 3 were discussed by Nguyen and Sampson (1985) in the context of examining the position of the set $M_{PQD}(p_1,p_2;q_1,q_2,\cdots,q_n)$ in

the class of all bivariate distributions. These cases were also discussed by Bhaskara Rao, Krishnaiah and Subramanyam (1987) in the spirit this paper is concerned with. Some measures of dependence, especially, Kendall's Tau, between pairs of random variables will be discussed in the light of the convexity property of the set $M_{PQD}(p_1,p_2;q_1,q_2,\cdots,q_n)$. The structure of this set will be exploited to examine properties of certain tests of independence. The affine property of certain measures of dependence makes it easy to evaluate the asymptotic powers of tests based on these measures of dependence.

2. ON EXTREME POINTS

Without loss of generality, assume that each of the numbers p_1, p_2, q_1, q_2, $\cdots$, q_n is positive. The following result characterizes members of $M_{PQD}(p_1,p_2;q_1,q_2,\cdots,q_n)$. This result is useful to develop an algebraic method for the extraction of all the extreme points of $M_{PQD}(p_1,p_2;q_1,q_2,\cdots,q_n)$. Some notation is in order. For any two real numbers a and b, $a \vee b$ denotes the maximum of a and b, $a \wedge b$ denotes the minimum of a and b.

Theorem 2. Let $P = (p_{ij}) \in M_{PQD}(p_1,p_2;q_1,q_2,\cdots,q_n)$. Then the numbers p_{11}, p_{12}, $\cdots$, p_{1n-1} satisfy the following inequalities.

$$
\begin{aligned}
& p_1q_1 \le p_{11} \le p_1 \wedge q_1 \\
& p_{11} \vee p_1(q_1+q_2) \le p_{11}+p_{12} \le p_1 \wedge (p_{11}+q_2) \\
& (p_{11}+p_{12}) \vee p_1(q_1+q_2+q_3) \le p_{11}+p_{12}+p_{13} \le p_1 \wedge (p_{11}+p_{12}+q_3) \qquad (2.1) \\
& \cdots \qquad \cdots \qquad \cdots \qquad \cdots \qquad \cdots \qquad \cdots \\
& (p_{11}+p_{12}+\cdots+p_{1n-2}) \vee p_1(q_1+q_2+\cdots+q_{n-1}) \le \\
& \qquad p_{11}+p_{12}+\cdots+p_{1n-1} \le p_1 \wedge (p_{11}+p_{12}+\cdots+p_{1n-2}+q_{n-1})
\end{aligned}
$$

Conversely, suppose $p_{11}, p_{12}, \cdots, p_{1n-1}$ are $(n-1)$ numbers satisfying the above inequalities. Let $p_{1n} = p_1 - (p_{11} + p_{12} + \cdots + p_{1n-1})$ and $p_{2j} = q_j - p_{1j}$, $j = 1,2,\cdots,n$. Then the matrix $P = (p_{ij}) \in M_{PQD}(p_1,p_2; q_1,q_2,\cdots,q_n)$.

Proof. Some comments are in order on the above result. Since the row and column sums of matrices $P = (p_{ij})$ in $M_{PQD}(p_1,p_2;q_1,q_2,\cdots,q_n)$ are fixed, knowledge of the $(n-1)$ numbers $p_{11}, p_{12}, \cdots, p_{1n-1}$ is sufficient. The remaining entries of P can be determined by appropriate subtractions. Coming to the proof, we observe that since P is positive quadrant dependent, $p_{11} + p_{12} + \cdots + p_{1i} \geq p_1(q_1 + q_2 + \cdots + q_i)$ for $i = 1,2,\cdots,n-1$. Since the entries of P are non-negative, $p_{11} + p_{12} + \quad + p_{1i} \geq p_{11} + p_{12} + \cdots + p_{1i-1}$ for $i = 2,3,\cdots,n-1$. These two sets of inequalities establish the validity of the inequalities on the left hand side of (2.1). In view of the marginality restrictions, it follows that $p_{11} + p_{12} + \cdots + p_{1i} \leq p_1$ and also $\leq p_{11} + p_{12} + \cdots + p_{1i-1} + q_i$ for $i = 1,2,\cdots,n-1$. Thus we see the validity of the inequalities on the right hand side of (2.1). Conversely, if $p_{11}, p_{12}, \cdots, p_{1n-1}$ satisfy the inequalities (2.1), it immediately follows that $p_{11} > 0$ and $p_{1j} \geq 0$ for $j = 2,3,\cdots,n-1$. From the inequalities on the right hand side of (2.1), it follows that each of p_{2j}, $j = 1,2,\cdots,n-1$ is non-negative. Observe also that p_{1n} is non-negative. It remains to be shown that p_{2n} is non-negative. Since $p_{2n} = q_n - p_{1n} = q_n - (p_1 - (p_{11} + p_{12} + \cdots + p_{1n-1})) = q_n - p_1 + (p_{11} + p_{12} + \cdots + p_{1n-1})$, it suffices to show that $p_{11} + p_{12} + \cdots + p_{1n-1} \geq p_1 - q_n$. From the last inequality of (2.1), it follows that $p_{11} + p_{12} + \cdots + p_{1n-1} \geq p_1(q_1 + q_2 + \cdots + q_{n-1})$ $= p_1(1 - q_n) = p_1 - p_1q_n \geq p_1 - q_n$. It is now clear that the matrix $P = (p_{ij})$ has the required marginal sums and that P is positive quadrant dependent. This completes the proof.

An algebraic method for the identification of the extreme points of $M_{PQD}(p_1,p_2,;q_1,q_2,\cdots,q_n)$

The following proposition in conjunction with Theorem 2 above would form the basis of the algebraic method we intend to develop.

Proposition 3. Let $a_1, a_2, \cdots, a_m$ and $b_1, b_2, \cdots, b_m$ be 2m fixed numbers. Let M be the set of all vectors $(x_1,x_2,\cdots,x_m)$ satisfying

$$
\begin{aligned}
& a_1 \le x_1 \le b_1 \\
& a_2 \le x_1 + x_2 \le b_2 \\
& a_3 \le x_1 + x_2 + x_3 \le b_3 \qquad (2.2) \\
& \cdots \qquad\qquad \cdots \qquad\qquad \cdots \\
& a_m \le x_1 + x_2 + \cdots + x_m \le b_m.
\end{aligned}
$$

Then M is a compact convex set and every extreme point $(x_1^*,x_2^*,\cdots,x_m^*)$ of M satisfies $x_1^* + x_2^* + \cdots + x_i^* = a_i$ or b_i for each $i = 1,2,\cdots,m$.

Proof. It is obvious that M is a compact convex set. Assume that M is non-empty. Let $(x_1^*,x_2^*,\ldots,x_m^*) \in M$. We will show that if $x_1^* + x_2^* + \ldots + x_i^*$ is neither equal to a_i nor to b_i for some i, then $(x_1^*,x_2^*,\ldots,x_m^*)$ is not an extreme point of M. Let i be the largest index in $\{1,2,\ldots,m\}$ such that $x_1^* + x_2^* + \ldots + x_i^*$ is neither equal to a_i nor to b_i. Let $x_1' = x_2'' = x_1^*$; $x_2' = x_2'' = x_2^*$; $\ldots$; $x_{i-1}' = x_{i-1}'' = x_{i-1}^*$. Solve the following equations

$$x_1' + x_2' + \cdots + x_i' = a_i$$

and

$$x_1'' + x_2'' + \cdots + x_i'' = b_i$$

for x_i' and x_i'', respectively. Then we can write $x_i^* = \lambda x_i' + (1-\lambda)x_i''$ for for some $0 < \lambda < 1$. Now solve the following equations

$$x'_1 + x'_2 + \cdots + x'_j = a_j$$

and

$$x''_1 + x''_2 + \cdots + x''_j = a_j$$

if $x^*_1 + x^*_2 + \cdots + x^*_j = a_j$

or the equations

$$x'_1 + x'_2 + \cdots + x'_j = b_j$$

and

$$x''_1 + x''_2 + \cdots + x''_j = b_j$$

if $x^*_1 + x^*_2 + \cdots + x^*_j = b_j$

for $j = i+1, i+2, \cdots, m$, successively. It is now clear that $(x'_1, x'_2, \cdots, x'_m)$ and $(x''_1, x''_2, \cdots, x''_m)$ are distinct, belong to M, and $(x^*_1, x^*_2, \cdots, x^*_m) = \lambda(x'_1, x'_2, \cdots, x'_m) + (1-\lambda)(x''_1, x''_2, \cdots, x''_m)$. Hence $(x^*_1, x^*_2, \cdots, x^*_m)$ cannot be an extreme point of M.

The extreme points of M are easy to find. Set the central expression in each of the inequalities (2.2) equal either to the quantity on the right or to the quantity on the left, and then solve the resultant system of equations in $x_1, x_2, \cdots, x_m$. These systems of equations are easy to solve recursively. A crude upper bound to the total number of extreme points of M is 2^m. The system of inequalities (2.1) bear a close resemblence to the system of inequalities (2.2). The presence of the symbols $\vee$ and $\wedge$ in the system (2.1) seem to complicate the problem of finding the extreme points of $M_{PQD}(p_1, p_2; q_1, q_2, \cdots, q_m)$. In order to reduce the system of inequalities (2.1) to the one of the form (2.2), we must find a way of getting rid of the symbols $\vee$ and $\wedge$ from (2.1). This can be achieved by splitting each inequality in the system (2.1)

into sub-inequalities, if necessary, using the subsequent inequalities to get rid of the symbols $\vee$ and $\wedge$. Eventually, we should be able to get systems of inequalities of the type (2.2) equivalent to (2.1). We will explain this method by working out some examples.

Example 1 The case when $n = 2$.

The determining inequalities (2.1) become

$$p_1q_1 \leq p_{11} \leq p_1 \wedge q_1.$$

There are only two extreme points of $M_{PQD}(p_1,p_2;q_1,q_2)$. If $p_1 \leq q_1$, these are given by

$$P_1 = \begin{bmatrix} p_1q_1 & p_1q_2 \\ p_2q_1 & p_2q_2 \end{bmatrix} \quad \text{and} \quad P_2 = \begin{bmatrix} p_1 & 0 \\ q_1-p_1 & q_2 \end{bmatrix}.$$

If $q_1 \leq p_1$, the extreme points are

$$P_1 = \begin{bmatrix} p_1q_1 & p_1q_2 \\ p_2q_1 & p_2q_2 \end{bmatrix} \quad \text{and} \quad P_2 = \begin{bmatrix} q_1 & p_1-q_1 \\ 0 & p_2 \end{bmatrix}.$$

Example 2 The case when $n = 3$.

The determining inequalities (2.1) become

$$p_1q_1 \leq p_{11} \leq p_1 \wedge q_1$$

and

$$p_{11} \vee p_1(q_1 + q_2) \leq p_{11} + p_{12} \leq p_1 \wedge (p_{11} + q_2).$$

Since there are only two variables, this system of inequalities

can be solved graphically. As an illustration, let $p_1 = 0.3$, $p_2 = 0.7$ and $q_1 = 0.5$, $q_2 = 0.3$, $q_3 = 0.2$. The inequalities (2.1) become

$$0.15 \leq p_{11} \leq 0.3$$

and

$$p_{11} \vee 0.24 \leq p_{11} + p_{12} \leq 0.3.$$

The maximum symbol $\vee$ in the second inequality above can be eliminated by splitting the first inequality into two parts : $0.15 \leq p_{11} \leq 0.24$ and $0.24 \leq p_{11} \leq 0.3$. The above system of inequalities is equivalent to the following two systems of inequalities.

$$0.15 \leq p_{11} \leq 0.24$$

(2.3) and

$$0.24 \leq p_{11} + p_{12} \leq 0.3;$$

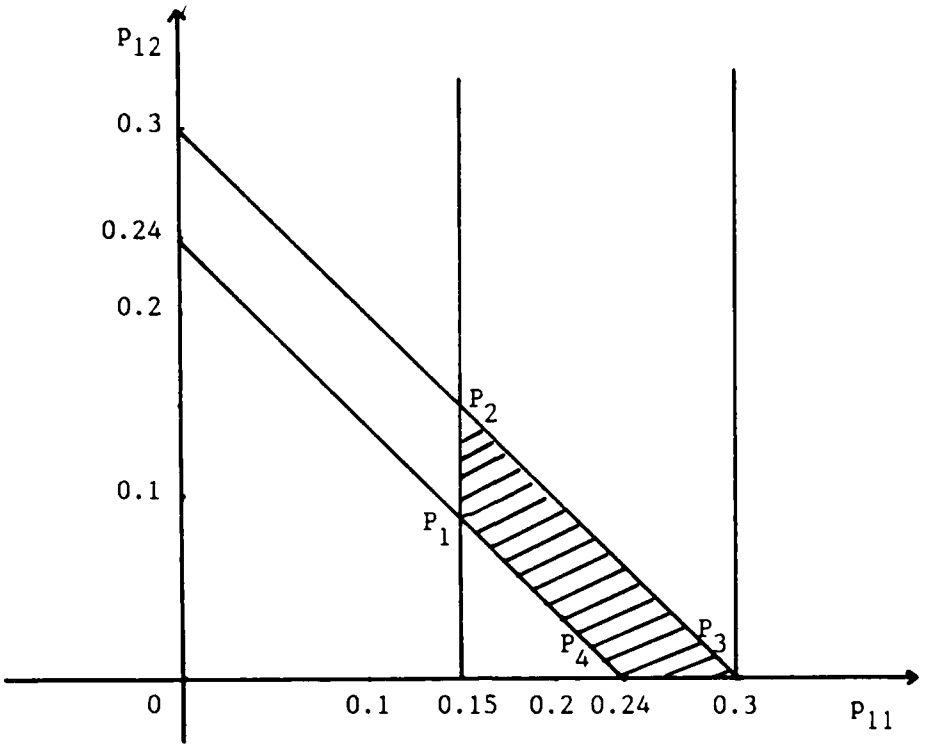
or

$$0.24 \leq p_{11} \leq 0.3$$

(2.4) and

$$p_{11} \leq p_{11} + p_{12} \leq 0.3$$

in the sense that if p_{11} and p_{12} satisfy the given system (2.1) of inequalities, then p_{11} and p_{12} satisfy either the system (2.3) or the system (2.4). The graph of these two systems of inequalities is given below.



Graph of the inequalities (2.3) and (2.4)

The extreme points of the convex set $M_{PQD}(0.3,0.7;0.5,0.3,0.2)$ can be read off directly from the above graph. These are given by

$$P_1 = \begin{bmatrix} 0.15 & 0.09 & 0.06 \\ 0.35 & 0.21 & 0.14 \end{bmatrix}; \qquad P_2 = \begin{bmatrix} 0.15 & 0.15 & 0.00 \\ 0.35 & 0.15 & 0.20 \end{bmatrix};$$

$$P_3 = \begin{bmatrix} 0.30 & 0.00 & 0.00 \\ 0.20 & 0.30 & 0.20 \end{bmatrix}; \qquad P_4 = \begin{bmatrix} 0.24 & 0.00 & 0.06 \\ 0.26 & 0.30 & 0.14 \end{bmatrix}.$$

One could use the algebraic method described following Proposition 3 to find the extreme points using the two systems of inequalities (2.3) and (2.4). After weeding out the duplicates and non-extreme points, one gets the same four matrices detailed above.

Example 3 The case when $n = 4$.

The determining inequalities (2.1) become

$$p_1 q_1 \leq p_{11} \leq p_1 \wedge q_1,$$

$$p_{11} \vee p_1(q_1 + q_2) \leq p_{11} + p_{12} \leq p_1 \wedge (p_{11} + q_2),$$

and

$$(p_{11} + p_{12}) \vee p_1(q_1 + q_2 + q_3) \leq p_{11} + p_{12} + p_{13} \leq p_1 \wedge (p_{11} + p_{12} + q_3).$$

For a specific illustration, take $p_1 = 0.6$, $p_2 = 0.4$ and $q_1 = 0.2$, $q_2 = 0.4$, $q_3 = 0.2$, $q_4 = 0.2$. The inequalities then reduce to

$$\begin{aligned} &0.12 \leq p_{11} \leq 0.2, \\ &0.36 \leq p_{11} + p_{12} \leq p_{11} + 0.4, \\ &\text{and} \\ &(p_{11} + p_{12}) \vee 0.48 \leq p_{11} + p_{12} + p_{13} \leq 0.6 \wedge (p_{11} + p_{12} + 0.2). \end{aligned} \tag{2.5}$$

The symbols $\vee$ and $\wedge$ are present only in the last inequality above. They can be eliminated by splitting the first two inequalities carefully. The system (2.5) is equivalent to the following three systems of inequalities.

$$0.12 \leq p_{11} \leq 0.2,$$

$$0.36 \leq p_{11} + p_{12} \leq 0.4,$$

and

$$0.48 \leq p_{11} + p_{12} + p_{13} \leq p_{11} + p_{12} + 0.2;$$

or

$$0.12 \leq p_{11} \leq 0.2,$$

$$0.4 \leq p_{11} + p_{12} \leq 0.48,$$

and

$$0.48 \leq p_{11} + p_{12} + p_{13} \leq 0.6;$$

or

$$0.12 \leq p_{11} \leq 0.2,$$

$$0.48 \leq p_{11} + p_{12} \leq p_{11} + 0.4,$$

and

$$p_{11} + p_{12} \leq p_{11} + p_{12} + p_{13} \leq 0.6.$$

For each system above, one can find its extreme points algebraically by setting the central expression in each inequality to either the quantity on the right or to the quantity on the left. After weeding out the repetitions and non-extreme points, one obtains the following as the collection of all extreme points of $M_{PQD}(0.6,0.4;0.2,0.4,0.2,0.2)$.

$$P_1 = \begin{bmatrix} 0.12 & 0.24 & 0.12 & 0.12 \\ 0.08 & 0.16 & 0.08 & 0.08 \end{bmatrix}; \quad P_2 = \begin{bmatrix} 0.12 & 0.24 & 0.20 & 0.04 \\ 0.08 & 0.16 & 0.00 & 0.16 \end{bmatrix};$$

$$P_3 = \begin{bmatrix} 0.12 & 0.28 & 0.20 & 0.00 \\ 0.08 & 0.12 & 0.00 & 0.20 \end{bmatrix}; \quad P_4 = \begin{bmatrix} 0.20 & 0.16 & 0.12 & 0.12 \\ 0.00 & 0.24 & 0.08 & 0.08 \end{bmatrix};$$

$$P_5 = \begin{bmatrix} 0.20 & 0.16 & 0.20 & 0.04 \\ 0.00 & 0.24 & 0.00 & 0.16 \end{bmatrix}; \quad P_6 = \begin{bmatrix} 0.20 & 0.20 & 0.08 & 0.12 \\ 0.00 & 0.20 & 0.12 & 0.18 \end{bmatrix};$$

$$P_7 = \begin{bmatrix} 0.20 & 0.20 & 0.20 & 0.00 \\ 0.00 & 0.20 & 0.00 & 0.20 \end{bmatrix}; \quad P_8 = \begin{bmatrix} 0.12 & 0.36 & 0.00 & 0.12 \\ 0.08 & 0.04 & 0.20 & 0.08 \end{bmatrix};$$

$$P_9 = \begin{bmatrix} 0.12 & 0.36 & 0.12 & 0.00 \\ 0.08 & 0.04 & 0.08 & 0.20 \end{bmatrix}; \quad P_{10} = \begin{bmatrix} 0.20 & 0.28 & 0.00 & 0.12 \\ 0.00 & 0.12 & 0.20 & 0.08 \end{bmatrix};$$

$$P_{11} = \begin{bmatrix} 0.20 & 0.28 & 0.12 & 0.00 \\ 0.00 & 0.12 & 0.08 & 0.20 \end{bmatrix}; \quad P_{12} = \begin{bmatrix} 0.12 & 0.40 & 0.00 & 0.08 \\ 0.08 & 0.00 & 0.20 & 0.12 \end{bmatrix};$$

$$P_{13} = \begin{bmatrix} 0.12 & 0.40 & 0.08 & 0.00 \\ 0.08 & 0.00 & 0.12 & 0.20 \end{bmatrix}; \quad P_{14} = \begin{bmatrix} 0.20 & .40 & 0.00 & 0.00 \\ 0.00 & 0.00 & 0.20 & 0.20 \end{bmatrix}$$

Remarks. For the general case of n, identification of extreme points can be carried out in a similar vein as outlined above.One can show that the joint distribution P_1 under which X and Y are independent is always an extreme point of $M_{PQD}(p_1,p_2;q_1,q_2,\cdots,q_n)$.

3. SOME APPLICATIONS

(1) On the affine property of certain measures of dependence

In the literature, there are several measures of dependence proposed to study the degree of dependence between two random variables. For details, one may refer to Agresti (1984, Chapter 9) or Goodman and Kruskal (1979). In this section, we examine the following problem. For any bivariate distribution P, let $\Delta(P)$ denote a measure of dependence proposed. If P_1 and P_2 are two bivariate distributions in $M_{PQD}(p_1,p_2;q_1,q_2,\cdots,q_n)$ and $0 \leq \lambda \leq 1$, is it true that

$$\Delta(\lambda P_1 + (1-\lambda)P_2) = \lambda\Delta(P_1) + (1-\lambda)\Delta(P_2)?$$

If it is so, $\Delta(\cdot)$ can rightly be called an affine measure of dependence on $M_{PQD}(p_1,p_2;q_1,q_2,\cdots,q_n)$.

Let $P = (p_{ij})$ be any bivariate distribution of order 2xn. Let

$$\Pi_c = 2[p_{11}(p_{22} + p_{23} + \cdots + p_{2n}) + p_{12}(p_{23} + p_{24} + \cdots + p_{2n}) + \cdots + p_{1n-1}p_{2n}]$$

and

$$\Pi_d = 2[p_{1n}\,(p_{21} + p_{22} + \cdots + p_{2n-1}) + p_{1n-1}(p_{21} + p_{22} + \cdots + p_{2n-2}) + \cdots + p_{12}p_{21}].$$

For any P in $M_{PQD}(p_1,p_2;q_1,q_2,\cdots,q_n)$, the following are some of the measures of dependence commonly used in the literature.

Kendall's Tau-b $\quad \tau_b(P) = (\Pi_c - \Pi_d)/[(1-p_1^2-p_2^2)(1-q_1^2-q_2^2-\cdots-q_n^2)]^{\frac{1}{2}}$

Somer's d (Y on X) $\quad d_{YX}(P) = (\Pi_c - \Pi_d)/(1-p_1^2-p_2^2)$

Somer's d (X on Y) $\quad d_{XY}(P) = (\Pi_c - \Pi_d)/(1-q_1^2-q_2^2-\cdots-q_n^2)$

Pearson's Correlation Coefficient $\quad \rho(P) = \mathrm{Cov}_P(X,Y)/[\mathrm{Var}_P(X)\ \mathrm{Var}_P(Y)]^{\frac{1}{2}}$

Spearman's rho $\quad \rho_S(P) = [\Sigma_{i=1}^2 \Sigma_{j=1}^n (r_i^X - 0.5)(r_j^Y - 0.5)p_{ij}] \div \{[\Sigma_{i=1}^2 (r_i^X - 0.5)^2 p_i][\Sigma_{j=1}^n (r_j^Y - 0.5)^2 q_j]\}^{\frac{1}{2}}$,

where $r_i^X = \Sigma_{k=1}^{i-1} p_k + p_i/2, \quad i = 1,2$

and $r_j^Y = \Sigma_{s=1}^{j-1} q_s + q_j/2, \quad j = 1,2,\cdots,n.$

Theorem 4 All the measures of dependence described above are affine on the set $M_{PQD}(p_1,p_2;q_1,q_2,\cdots,q_n)$.

Proof. That the measures $\rho(\cdot)$ and $\rho_S(\cdot)$ are affine is obvious since these measures are linear functions of the joint probabilities and the marginals are fixed. For the affine property of $\tau_b(\cdot)$, $d_{YX}(\cdot)$ and $d_{XY}(\cdot)$, it suffices to show that $\Pi_c - \Pi_d$ is a linear function of the joint probabilities. By substituting $p_{2j} = q_j - p_{1j}$, $j = 1,2,\cdots,n$, one can easily verify that

$$\Pi_c - \Pi_d = 2\,\Sigma_{j=1}^{n-1} p_{1j} - 2\,\Sigma_{j=1}^{n-1} (q_1+q_2+\cdots+q_j)(p_{1j}+p_{1j+1}).$$

This completes the proof.

Remarks In view of the above theorem, it suffices to compute the measure of dependence for the extreme point distributions of $M_{PQD}(p_1,p_2;q_1,q_2,\cdots,q_n)$ for any of the measures of dependence described above. The measure of dependence for any distribution P in $M_{PQD}(p_1,p_2;q_1,q_2,\cdots,q_n)$ is a convex combination of the corresponding measures of dependence for the extreme point distributions in $M_{PQD}(p_1,p_2;q_1,q_2,\cdots,q_n)$. we will make use of this property in computing asymptotic power of certain tests of independence.

(2) Computation of the power function of tests of independence

Suppose X and Y are two random variables such that X takes only two values, say, 1 and 2, and Y takes n values, say, $1,2,\cdots,n$. Assume that the marginal distributions of X and Y are known given by p_1,p_2 and $q_1,q_2,\cdots,q_n$, respectively. The joint distribution $P = (p_{ij})$ of X and Y is unknown but known to be positive quadrant dependent. Suppose we wish to test the validity of the hypothesis

H_o : X and Y are independent

against

H_1 : X and Y are strictly positive quadrant dependent

based on N independent realizations $(X_1,Y_1), (X_2,Y_2), \cdots, (X_N,Y_N)$ of (X,Y). The null hypothesis H_o is simple and the alternative H_1 is composite. Let $P_1 = (p_iq_j)$. Then H_o and H_1 can be written equivalently as

$$H_o : P = P_1$$

$$H_1 : P \neq P_1, \quad P \in M_{PQD}(p_1,p_2;q_1,q_2,\cdots,q_n).$$

If T is any test proposed for H_o against H_1, one could use the substance of Theorem 1 to give a simple expression for the power function of T. Let

$$\beta_T(P) = \Pr(T \text{ rejects } H_o \mid \text{The joint distribution of } X \text{ and } Y \text{ is } P),$$

$$\text{for } P \in M_{PQD}(p_1,p_2;q_1,q_2,\cdots,q_n)$$

be the power function of T. Obviously, $\beta_T(P_1)$ is the size of the test T. We explain how $\beta_T(P)$ can be simplified when $n = 2$. Let $P_1 = (p_i q_j)$ and P_2 be the extreme points of $M_{PQD}(p_1,p_2;q_1,q_2)$. Let $P \in M_{PQD}(p_1,p_2;q_1,q_2)$. Then we can write $P = \lambda P_1 + (1-\lambda)P_2$ for some $0 \leq \lambda \leq 1$. If the joint distribution of X and Y is P, we denote the joint distribution of the random sample $(X_1,Y_1), (X_2,Y_2), \cdots, (X_N,Y_N)$ by P^N. (P^N does not mean that the matrix P is multiplied by itself N times. P^N is the product probability measure.) Then

$$P^N = [\lambda P_1 + (1-\lambda)P_2]^N = \Sigma_{r=o}^N \binom{N}{r} \lambda^r (1-\lambda)^{N-r}(P_1^r \bullet P_2^{N-r}),$$

where $P_1^r \bullet P_2^{N-r}$ is the joint distribution of $(X_1,Y_1), (X_2,Y_2), \cdots, (X_N,Y_N)$ with $(X_1,Y_1), (X_2,Y_2), \cdots, (X_N,Y_N)$ independently distributed, $(X_1,Y_1), (X_2,Y_2), \cdots, (X_r,Y_r)$ identically distributed with common distribution P_1, and $(X_{r+1},Y_{r+1}), (X_{r+2},Y_{r+2}), \cdots, (X_N,Y_N)$ identically distributed with common distribution P_2. We can extend the definition of $\beta_T(P)$ to encompass this situation. Let

$$\beta_T(P_1^r \bullet P_2^{N-r}) = \Pr(T \text{ rejects } H_o \mid (X_1,Y_1), (X_2,Y_2), \cdots, (X_r,Y_r) \text{ have the common distribution } P_1 \text{ and } (X_{r+1},Y_{r+1}), (X_{r+2},Y_{r+2}), \cdots, (X_N,Y_N) \text{ have the common distribution } P_2),$$

$$\text{for } r = 0,1,2,\cdots,N.$$

In this new notation, $\beta_T(P) = \beta_T(P^N)$.

The following result is obvious.

Theorem 5 For any P in $M_{PQD}(p_1,p_2;q_1,q_2)$,

$$\beta_T(P) = \Sigma_{r=o}^{N} \binom{N}{r} \lambda^r (1-\lambda)^{N-r} \beta_T(P_1^r \bullet P_2^{N-r}),$$

where $P = \lambda P_1 + (1-\lambda)P_2$ with $0 \leq \lambda \leq 1$.

Remarks. In view of the above result, it suffices to compute $\beta_T(P_1^r \bullet P_2^{N-r})$ for $r = 0,1,2,\cdots,N$. $\beta_T(P)$ is a convex combination of these numbers for any P in $M_{PQD}(p_1,p_2;q_1,q_2)$. This formula is also useful if one wishes to compare the power functions of any two tests. This theorem has been exploited by Bhaskara Rao, Krishnaiah and Subramanyam (1987) to compare the performance of some tests in the case of 2x3 bivariate distributions in small samples. For small sample sizes N, computation of power functions of tests and comparison of power functions of tests can be carried out effectively using Theorem 5.

We now wish to emphasize that the affine property of certain measures of dependence also helps in evaluating the power function asymptotically for some tests built on these measures of dependence. As an illustration, consider the test based on Kendall's Tau-b. Let

$$O_{ij} = \#\{(X_r,Y_r)'s;\ X_r = i \text{ and } Y_r = j\},$$

for $i = 1,2$ and $j = 1,2$.

The data $(X_1,Y_1), (X_2,Y_2), \cdots, (X_N,Y_N)$ can be summarized in the form of a contingency table as follows.

X \ Y	1	2
1	O_{11}	O_{12}
2	O_{21}	O_{22}

An estimator of Kendall's Tau-b is given by

$$\hat{\tau}_{b,N} = (2/N^2)(O_{11}O_{22} - O_{12}O_{21})/[(1-p_1^2-p_2^2)(1-q_1^2-q_2^2)]^{\frac{1}{2}}.$$

Test based on Kendall's Tau-b

T : Reject H_o if and only if $\hat{\tau}_{b,N} > c$,

where c is the critical value of the test T

which depends on the given level of significance.

We now evaluate the power function of the test T asymptotically. Assume, without loss of generality, that $p_1 \leq q_1$. The extreme points of $M_{PQD}(p_1,p_2;q_1,q_2)$ are

$$P_1 = \begin{bmatrix} p_1q_1 & p_1q_2 \\ p_2q_1 & p_2q_2 \end{bmatrix} \quad \text{and} \quad P_2 = \begin{bmatrix} p_1 & 0 \\ q_1-p_1 & q_2 \end{bmatrix}.$$

Note that $\tau_b(P_1) = 0$ and $\tau_b(P_2) = 2p_1q_1/[(1-p_1^2-p_2^2)(1-q_1^2-q_2^2)]^{\frac{1}{2}} = \delta$, say. Then for any P in $M_{PQD}(p_1,p_2;q_1,q_2)$ with $P = \lambda_P P_1 + (1-\lambda_P)P_2$ and $0 \leq \lambda_P \leq 1$, it follows that $\tau_b(P) = (1-\lambda_P)\delta$.

The asymptotic power function of T has the following structure.

Theorem 6 Let $P \in M_{PQD}(p_1, p_2; q_1, q_2)$.

(i) If $c < 0$, then $\lim_{N \to \infty} \beta_T(P) = 0$.

(ii) If $c > 0$, then $\lim_{N \to \infty} \beta_T(P) = 1$ if $(1-\lambda_P)\delta > c$

$= 0$ if $(1-\lambda_P)\delta < c$.

Proof. Observe that by the Strong Law of Large Numbers

$\hat{\tau}_{b,N}$ converges to $\tau_b(P) = (1-\lambda_P)\delta$

almost surely as $N \to \infty$ under the joint distribution P of X and Y. From this, the desired conclusion follows.

Remarks. It is heartening to note that the size of the test T converges to 0 as $N \to \infty$ for $c > 0$. For $P \neq P_1$, one would like to have the power $\beta_T(P)$ to converge to 1 as $N \to \infty$. As is to be expected, this is not the case. The hypotheses H_o and H_1 are not well separated. If the joint distribution P is very close to P_1, the power $\beta_T(P)$ converges to 0 as $N \to \infty$. However, if there is reason to believe that the bivariate distributions under the alternative hypothesis are close to P_2, then the test T has the desired properties mentioned above. More specifically, let $0 < d < 1$ be fixed. Let the hypotheses be

$$H_o : P = P_1$$

and

$$H_1 : P = \lambda P_1 + (1-\lambda)P_2 \text{ with } 0 < \lambda \leq d.$$

The hypotheses H_0 and H_1 are well separated. We can choose a suitable c which figures in the description of the test T so that we will have

$$\lim_{N \to \infty} \beta_T(P_1) = 0$$

and

$$\lim_{N \to \infty} \beta_T(P) = 1$$

for every P under H_1.

Remarks. Computation of the asymptotic power of tests based on affine measures of dependence can be carried out in the same vein as above for any general n.

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The set of all bivariate distributions with support contained in $\{(i,j);\ i=1,2$ and $j = 1,2,\ldots,n\}$ which are positive quadrant dependent is a convex set. In this paper, an algebraic method is presented for the enumeration of all extreme points of the convex set. Certain measures of dependence, including Kendall's tau, are shown to be affine functions of convex set. This property of being affine helps us to evaluate the asymptotic power of tests based on these measures of dependence for testing the hypothesis of independence against strict positive quadrant dependence.

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